Problem 1.7

Two long, cylindrical conductors of radii a_1 and a_2 are parallel and separated by a distance d, which is large compared with either radius. Show that the capacitance per unit length is given approximately by

$$C \simeq \pi\epsilon_0 \left(\ln \frac{d}{a}\right)^{-1}$$

where a is the geometrical mean of the two radii.

Approximately what gauge wire (state diameter in millimeters) would be necessary to make a two-wire transmission line with a capacitance of 1.2×10^{-11} F/m if the separation of the wires was 0.5 cm? 1.5 cm? 5.0 cm?

Solution

The governing equations of the electric field are Gauss's law and Faraday's law. In the context of electrostatics in vacuum they are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\nabla \times \mathbf{E} = \mathbf{0}.$$

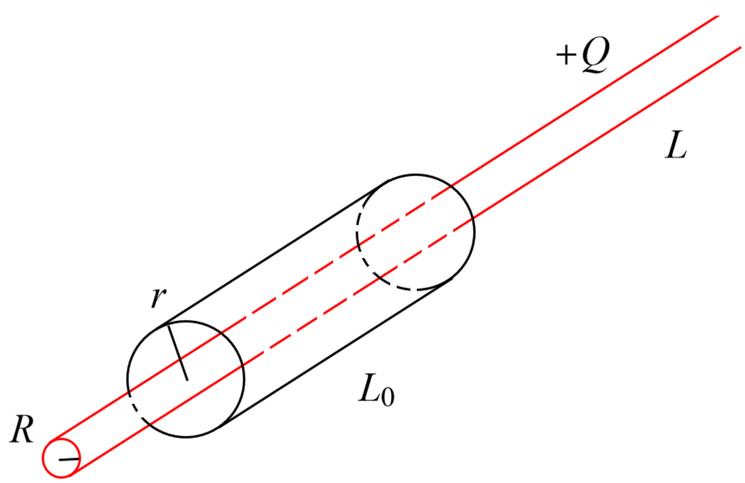
This second equation implies the existence of a potential function $-\Phi$ that satisfies

$$\mathbf{E} = \nabla(-\Phi) = -\nabla\Phi. \tag{2}$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). Assuming that empty space, or vacuum, separates the conductors, the capacitance is

$$C = \frac{Q}{\Phi}.$$

Note that Φ is interpreted as the work it takes to move a positive unit charge from the low-potential conductor to the high-potential conductor. Begin by finding the electric field of a long charged cylinder with radius R and length L.



Integrate both sides of equation (1) over the volume V_0 enclosed by the black Gaussian surface shown above.

$$\iiint_{V_0} \nabla \cdot \mathbf{E}\, dV = \iiint_{V_0} \frac{\rho}{\epsilon_0}\, dV$$

Using the divergence theorem on the left side results in the integral form of Gauss's law.

$$\oiint_{S_0} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \iiint_{V_0} \rho\, dV$$

The volume integral on the right is the total charge enclosed; since the charge is evenly distributed on the cylinder, the total enclosed charge is $\sigma(2\pi R L_0)$, where $\sigma = Q/(2\pi R L)$.

$$\oiint_{S_0} \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\epsilon_0} \left(\frac{Q}{L} L_0\right)$$

Because of the symmetry with respect to the cylinder's axis, the electric field is entirely radial: $\mathbf{E} = E_r \hat{\mathbf{r}}$.

$$\oiint_{S_0} E_r \hat{\mathbf{r}} \cdot (\hat{\mathbf{r}}\, dS) = \frac{1}{\epsilon_0} \left(\frac{Q}{L} L_0\right)$$

E_r is constant on the Gaussian surface S_0, so it can be pulled in front of the integral.

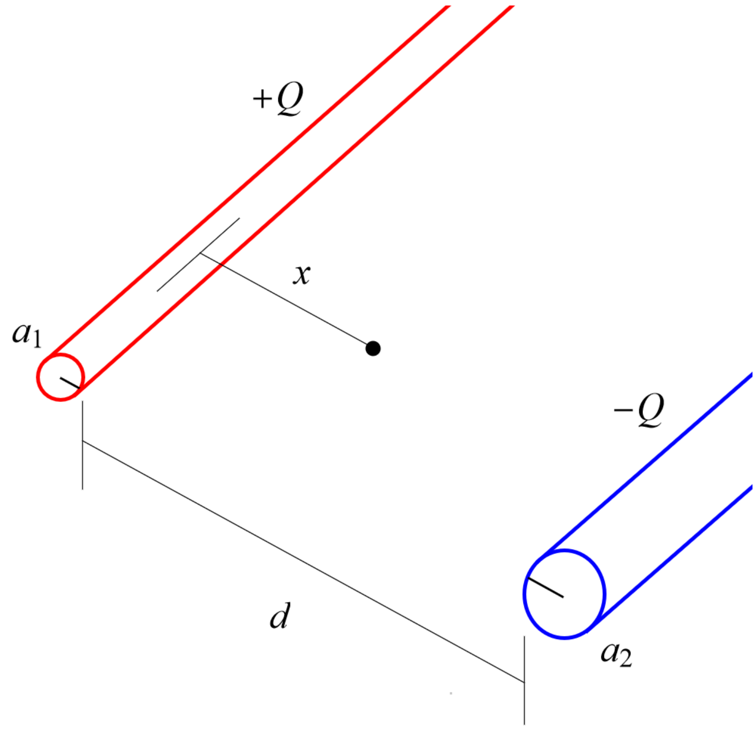
$$E_r \oiint_{S_0} dS = \frac{1}{\epsilon_0} \left(\frac{Q}{L} L_0\right)$$

Solve for E_r, the electric field magnitude.

$$E_r(2\pi r L_0) = \frac{Q L_0}{\epsilon_0 L}$$

$$E_r = \frac{Q}{2\pi \epsilon_0 r L}$$

In this problem there are two long parallel cylinders with radii, a_1 and a_2, and charges, $+Q$ and $-Q$, respectively, so the plan is to apply the principle of superposition to obtain the total electric field at a point between the cylinders in the same plane.



Add the electric fields from each of the cylinders vectorially to obtain the total electric field.

$$\begin{aligned}\mathbf{E} &= \frac{+Q}{2\pi\epsilon_0 x L}\hat{\mathbf{x}} + \frac{-Q}{2\pi\epsilon_0(a_1+d+a_2-x)L}(-\hat{\mathbf{x}}) \\ &= \frac{Q}{2\pi\epsilon_0 L}\left(\frac{1}{x} + \frac{1}{a_1+d+a_2-x}\right)\hat{\mathbf{x}} \\ &= \frac{Q}{2\pi\epsilon_0 L}\left(\frac{1}{x} - \frac{1}{x-a_1-d-a_2}\right)\hat{\mathbf{x}}\end{aligned}$$

Since what we need is the potential difference between the cylinders, we consider the x-component of equation (2) for $a_1 < x < a_1 + d$.

$$\mathbf{E} = -\nabla\Phi \quad \to \quad E_x = -\frac{d\Phi}{dx} \quad \to \quad \Phi(x) = -\int E_x \, dx$$

Φ is the work required to move a positive unit charge from the (low-potential) $-Q$ cylinder to the (high-potential) $+Q$ cylinder, so the limits of integration go from $a_1 + d$ to a_1.

$$\begin{aligned}\Phi &= -\int_{a_1+d}^{a_1} \frac{Q}{2\pi\epsilon_0 L}\left(\frac{1}{x} - \frac{1}{x-a_1-d-a_2}\right)dx \\ &= \frac{Q}{2\pi\epsilon_0 L}\int_{a_1}^{a_1+d}\left(\frac{1}{x} - \frac{1}{x-a_1-d-a_2}\right)dx\end{aligned}$$

Evaluate the integral and simplify the result.

$$\begin{aligned}
\Phi &= \frac{Q}{2\pi\epsilon_0 L}\left(\int_{a_1}^{a_1+d}\frac{dx}{x} - \int_{a_1}^{a_1+d}\frac{dx}{x-a_1-d-a_2}\right) \\
&= \frac{Q}{2\pi\epsilon_0 L}\left(\ln|x|\Big|_{a_1}^{a_1+d} - \ln|x-a_1-d-a_2|\Big|_{a_1}^{a_1+d}\right) \\
&= \frac{Q}{2\pi\epsilon_0 L}\left(\ln|a_1+d| - \ln|a_1| - \ln|-a_2| + \ln|-d-a_2|\right) \\
&= \frac{Q}{2\pi\epsilon_0 L}\Big[\ln(a_1+d) - \ln a_1 - \ln a_2 + \ln(d+a_2)\Big] \\
&= \frac{Q}{2\pi\epsilon_0 L}\ln\frac{(d+a_1)(d+a_2)}{a_1 a_2} \\
&= \frac{Q}{2\pi\epsilon_0 L}\ln\left[\frac{d^2}{a_1 a_2}\left(1+\frac{a_1}{d}\right)\left(1+\frac{a_2}{d}\right)\right]
\end{aligned}$$

Since $d \gg a_1$ and $d \gg a_2$, these terms in parentheses are very close to 1.

$$\Phi \approx \frac{Q}{2\pi\epsilon_0 L}\ln\left[\frac{d^2}{a_1 a_2}(1)(1)\right] = \frac{Q}{2\pi\epsilon_0 L}\ln\frac{d^2}{a_1 a_2} = \frac{Q}{\pi\epsilon_0 L}\ln\frac{d}{\sqrt{a_1 a_2}}$$

This quantity $a = \sqrt{a_1 a_2}$ is called the geometric mean of a_1 and a_2.

$$\Phi \approx \frac{Q}{\pi\epsilon_0 L}\ln\frac{d}{a}$$

The capacitance of these two parallel cylinders is then

$$C = \frac{Q}{\Phi} \approx \frac{Q}{\frac{Q}{\pi\epsilon_0 L}\ln\frac{d}{a}} = \frac{\pi\epsilon_0 L}{\ln\frac{d}{a}}.$$

Therefore, the capacitance per unit length is

$$\boxed{\frac{C}{L} \approx \pi\epsilon_0\left(\ln\frac{d}{a}\right)^{-1}.}$$

Since we want to determine the wire diameter, solve this formula for $2a$.

$$\ln\frac{d}{a} \approx \frac{\pi\epsilon_0}{\frac{C}{L}}$$

$$-\ln\frac{a}{d} \approx \frac{\pi\epsilon_0}{\frac{C}{L}}$$

$$\frac{a}{d} \approx \exp\left(-\frac{\pi\epsilon_0}{\frac{C}{L}}\right)$$

Multiply both sides by $2d$.

$$2a \approx 2d\exp\left(-\frac{\pi\epsilon_0}{\frac{C}{L}}\right)$$

Assuming a capacitance per unit length of 1.2×10^{-11} F/m, the diameter is about

$$2a \approx 2d \exp\left(-\frac{\pi\epsilon_0}{\frac{C}{L}}\right) \approx 2(5 \text{ mm}) \exp\left[-\frac{\pi(8.854 \times 10^{-12})}{1.2 \times 10^{-11}}\right] \approx 1 \text{ mm}$$

if the separation distance is $d = 0.5$ cm $= 5$ mm, the diameter is about

$$2a \approx 2d \exp\left(-\frac{\pi\epsilon_0}{\frac{C}{L}}\right) \approx 2(15 \text{ mm}) \exp\left[-\frac{\pi(8.854 \times 10^{-12})}{1.2 \times 10^{-11}}\right] \approx 3.0 \text{ mm}$$

if the separation distance is $d = 1.5$ cm $= 15$ mm, and the diameter is about

$$2a \approx 2d \exp\left(-\frac{\pi\epsilon_0}{\frac{C}{L}}\right) \approx 2(50 \text{ mm}) \exp\left[-\frac{\pi(8.854 \times 10^{-12})}{1.2 \times 10^{-11}}\right] \approx 9.8 \text{ mm}$$

if the separation distance is $d = 5.0$ cm $= 50$ mm.